

THE OUTERCOARSENESS OF THE  $n$ -CUBE

ALEX FINK AND RICHARD K. GUY

ABSTRACT. Guy and Nowakowski showed that the outercoarseness of the  $n$ -cube was, for sufficiently large  $n$ , at least 0.96 of its maximum possible value,  $n \cdot 2^{n-4}$ . Here we give some exact results, including that the maximum is attained for all  $n \geq 24$ . We construct explicit partitions of the edges of the cube attaining this maximum in which each part is a *tepee*, namely, the three-cube with a vertex and a non-incident edge deleted. Its vertices and those of the cube are given binary labels, which we often write in octal (base 8) or hexadecimal (base 16) notation.

## 1. INTRODUCTION

The *coarseness* of a graph is the maximum possible number of components in an edge-partition of the graph with each part a non-planar graph. Interest in coarseness, a one of several quantities one might associate to a graph to measure its deviation from planarity, dates from Erdős's introduction of the concept. One application is to printing of circuits: if one can print conducting material onto insulating sheets, with contact between the sheets at and only at a set of points serving as the vertices, then the coarseness of a graph is the number of sheets required to realise it [6]. Bounds or exact values have been computed for the coarseness of several frequently studied families of graphs, among them the  $n$ -dimensional cube  $Q_n$  [5, 4].

Similarly, the *outercoarseness*  $\xi_o(G)$  of a graph  $G$  is the maximum number of parts in an edge partition with each part non-outerplanar. An *outerplanar graph* is a planar graph for which there is an imbedding in the plane with all its vertices on the boundary of a single cell. Bounds for the outercoarseness of the cube  $Q_n$  were first announced without proof in [1], with the proofs appearing in the follow-up article [2]. The upper bound given in these works was

$$(1.1) \quad \xi_o(Q_n) \leq n \cdot 2^{n-4},$$

a consequence of a simple edge-counting argument, while the lower bound was  $(0.96n - 1.15)2^{n-4} < \xi_o(Q_n)$  for  $n$  not equal to 2, 5, 6 or 9.

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The present paper closes the gap between the above bounds for all  $n \geq 24$ , as well as certain lesser values of  $n$ , by exhibiting edge-partitions that show that the upper bound (1.1) is in fact attainable. We also give improved bounds in some smaller cases.

Halin [3] has given the analog, for outerplanar graphs, of Kuratowski's theorem for planar graphs:

**Theorem 1.1** (Halin). *A graph is outerplanar just if it does not contain a subgraph homeomorphic to the complete graph  $K_4$  or to the complete bipartite graph  $K_{2,3}$ .*

When we refer to a product  $G \times H$  of two graphs we mean the *Cartesian product*, in which two vertices  $(v, w)$  and  $(v', w')$  are joined by an edge if and only if either  $v = v'$  and  $w$  is adjacent to  $w'$  in  $H$ , or  $v$  is adjacent to  $v'$  in  $G$  and  $w = w'$ .

For example, the  $n$ -cube,  $Q_n$ , is the  $n$ -fold Cartesian product of  $Q_1 = K_2$ , and is therefore a graph on  $2^n$  vertices. If we give the names 0 and 1 to the two vertices of  $Q_1$ , the vertices of  $Q_n$  are thereby labelled with the  $n$ -digit binary numbers. The vertex labels form a  $n$ -dimensional vector space over  $\mathbb{F}_2$ , and the symbols “+” and “span” will refer to operations inside this vector space. The operation + is also known as *bitwise xor*, or *nim-sum*. After this introduction, we will begin to write the binary labels in octal or hexadecimal notation. Although this labelling may be novel, it is purely a space-saving measure, compressing three or four coordinates of the vertex label to a single symbol.

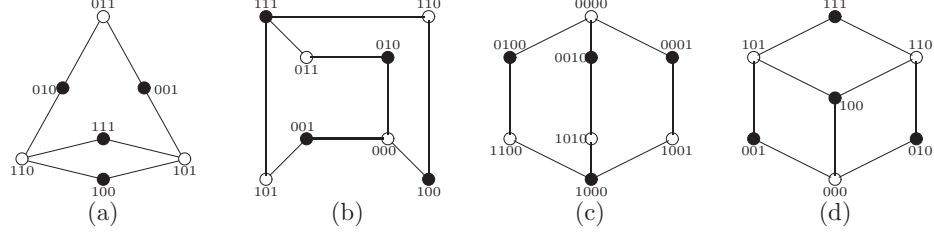
The  $n$ -cube is regular with valence  $n$ . The  $n \cdot 2^{n-1}$  edges connect pairs of vertices whose labels differ in just one binary digit. Of the edges there are  $2^{n-1}$  in each of  $n$  different *directions*, or having  $n$  different *colors*, according to which of the  $n$  Cartesian factors  $Q_1$  give rise to them. The  $n$ -cube is bipartite: the parts are the set of *odious* vertices and the set of *evil* vertices, the digit-sums of whose labels are respectively odd and even, and which are respectively represented by small black and white circles in Figure 1.

The smallest non-outerplanar subgraphs of  $Q_n$  are shown in Figure 1, where (a), (b), (c) are homeomorphs of  $K_{2,3}$  and (d) is a homeomorph of  $K_4$ . The vertex labels are in binary. Only (a) has as few as 8 edges, and it is easy to see that no such graph can have fewer edges. So (a) will be the graph of our choice, which we call a *tepee*.

This immediately gives us the upper bound (1.1),

$$\xi_o(Q_n) \leq n \cdot 2^{n-4}.$$

We set some terminology for the anatomy of a tepee. Let  $e_0 = 00 \dots 01$ ,  $e_1 = 00 \dots 10$ ,  $\dots$ ,  $e_{n-1} = 10 \dots 00$  be the standard basis of the  $\mathbb{F}_2$ -vector space whose elements are the vertices of  $Q_n$ , so that the neighbours of a vertex  $v$  are all vertices of form  $v + e_i$  with  $0 \leq i \leq n-1$ . Then a *tepee* in

FIGURE 1. Non-outerplanar subgraphs of  $Q_n$ 

orientation  $ijk$  contains the edges

$$\begin{aligned}
 (1.2) \quad & (v, v + e_i), & (v, v + e_j), \\
 & (v + e_i + e_k, v + e_i), & (v + e_j + e_k, v + e_j), \\
 & (v + e_i + e_k, v + e_k), & (v + e_j + e_k, v + e_k), \\
 & (v + e_i + e_k, v + e_i + e_j + e_k), & (v + e_j + e_k, v + e_i + e_j + e_k)
 \end{aligned}$$

for some vertex  $v$ , which we will call the *tip* of the tepee. The tepee's trivalent vertices are  $v + e_i + e_k$  and  $v + e_j + e_k$ , and its remaining vertices are bivalent. A tepee subgraph of a cube is completely specified by its tip and orientation, and this specification is unique up to exchange of  $i$  and  $j$ . For example, the tepee in Figure 1(a) has tip 011, and orientation 012 (or equally 102). When checking the orientations, bear in mind that we have labelled the bits *from right to left*, e.g., vertex 011 means  $0e_2 + 1e_1 + 1e_0$ !

## 2. SMALL VALUES OF $n$

The bound (1.1) can't be attained for small  $n$ . For  $n = 1$  and  $2$ ,  $Q_n$  is outerplanar, so that  $\xi_o(Q_1) = \xi_o(Q_2) = 0$ . On the other hand  $Q_3$  is not outerplanar, but has only 12 edges, so that  $\xi_o(Q_3) = 1$ . Although  $Q_4$  has 32 edges, the bound of 4 can't be attained since the graph of Figure 1(a) has two trivalent vertices, so that some of the edges of the 4-valent graph  $Q_4$  can't be used. That  $\xi_o(Q_4) = 3$  may be seen from Figure 2 whose vertex labels are written in octal. It shows the decomposition of  $Q_4$  into two tepees in orientation 120 with tips 02 and 12, and a copy of Figure 1(c): there are 7 unused edges, indicated by dotted lines.



Four of them,  $(1y, 3y)$ ,  $y = 1, 3, 5, 7$ , together with the edges between their endpoints in the two  $Q_4$ , form a  $Q_3$ , which is non-outerplanar. The four pairs,  $(1x, 3x)$  and  $(0y, 2y)$ ,  $y = x+1$ , together with the edges in the centre and right of Figure 3 in directions 0 and 3 form tepees. With the tepees from the two  $Q_4$ , we have  $1 + 4 + (2 \times 2) = 9$  non-outerplanar graphs and

$$\xi_o(Q_5) = 9.$$

#### 4. THE 6-CUBE

If  $Q_6$  had an exact packing with 24 tepees, every vertex of  $Q_6$  would have to be occupied either by three bivalent or two trivalent tepee vertices. Thus the tepees would be arranged into rings, where neighboring tepees in each ring share a trivalent vertex.

A program was run to find packings with rings of tepees, which found that the maximal packing of this sort uses 16 tepees. This establishes  $\xi_o(Q_6) < 24$ . Here is a packing of 16 tepees in four rings of four, with vertex labels given in two-digit octal, and each ring described by one column:

orient <sup>n</sup>	tip	orient <sup>n</sup>	tip	orient <sup>n</sup>	tip	orient <sup>n</sup>	tip
012	06	012	53	012	17	012	42
345	50	345	05	345	41	345	14
012	35	012	60	012	24	012	71
345	63	345	36	345	72	345	27

More compactly, this arrangement uses 8 copies of the half-ring consisting of one tepee in orientation 012 with tip 06 and one in orientation 345 with tip 50, translating it by the elements of  $\text{span}\{11, 22, 44\}$ .

We also wrote a second, unclever, program which placed tepees unrestrictedly. It quickly found our current best known solutions with 21 tepees; it seemed to plateau there, though we never let it run to exhaustion. The table below exhibits one of its packings of 21 tepees in  $Q_6$ . The source code of both of these programs, with a brief explanation of their output formats, can be found at <http://www.maths.qmul.ac.uk/~fink/outercoarseness/Q6.html>.

orient <sup>n</sup>	tip	orient <sup>n</sup>	tip	orient <sup>n</sup>	tip
012	05	145	44	123	33
345	50	053	44	150	74
342	11	054	47	231	44
054	41	150	50	123	55
032	12	124	30	024	50
054	42	450	17	123	72
345	53	231	24	123	73

The resulting bounds are

$$21 \leq \xi_o(Q_6) \leq 23.$$

## 5. THE 7-CUBE

We can show that  $48 \leq \xi_o(Q_7) \leq 54$ .

Taking the upper bound first, every non-outerplanar subgraph of a cube has at least 7 vertices with valence  $\geq 2$ ; observe that this is true of the minimal such graphs in Figure 1. Suppose that  $t$  non-outerplanar graphs are packed in  $Q_7$ . Each vertex of  $Q_7$  can only accommodate three vertices of valence  $\geq 2$ , proving  $t \leq \lfloor 3 \cdot 2^7 / 7 \rfloor = 54$ .

The lower bound is attained by the following packing of 48 tepees. Writing in octal, place 32 tepees with their tips two each at the sixteen vertices of  $\text{span}\{011, 022, 044, 100\}$ , one in orientation 012 and one in orientation 345 for each tip location. Place the remaining sixteen as follows:

- four in orientation 036 with tips  $003 + \text{span}\{022, 044\}$ ;
- four in orientation 146 with tips  $003 + \text{span}\{011, 044\}$ ;
- four in orientation 256 with tips  $003 + \text{span}\{011, 022\}$ ;
- and four in orientation 256 with tips  $\text{span}\{011, 022\}$ .

6. VALUES OF  $n$  ATTAINING THE UPPER BOUND

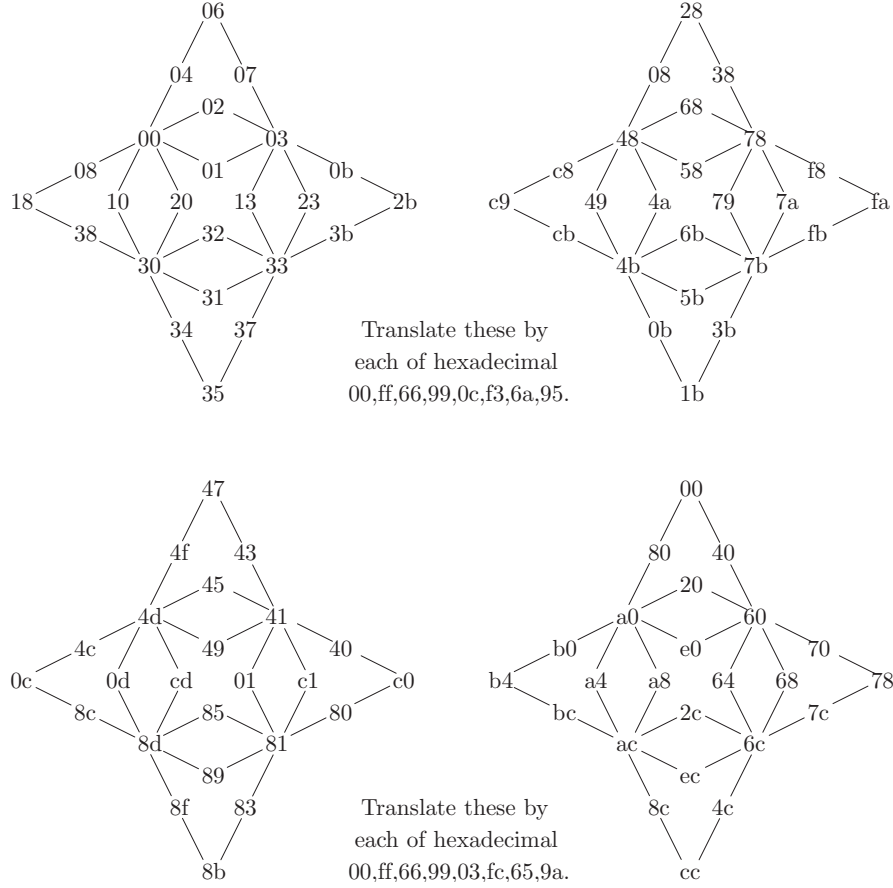
Here we construct exact packings demonstrating that  $\xi_o(Q_n)$  attains its maximum value  $n \cdot 2^{n-4}$  for all  $n \geq 8$  other than  $n = 9, 10, 11, 14, 15, 19, 23$ , which we cannot address at present. These constructions are based on the following elements:

- (1) Exact tepee packings of  $Q_8$ ,  $Q_{12}$ ,  $Q_4 \times K_{1,2}$  and  $Q_8 \times K_{1,2}$ . By taking products of these we obtain all graphs  $Q_{4k} \times (Q_4 \times K_{1,2})^m$  for integers  $k, m \geq 0$ , aside from  $Q_4$ .
- (2) A partial arrangement of tepees in  $Q_5$  which, together with packings of  $G$  and  $G \times K_{1,2}$ , provides a packing of  $G \times Q_5$  for any graph  $G$ . In view of item 1, iteration of this construction yields an exact tepee packing of every cube  $Q_{4k+9m}$  for integers  $k, m \geq 0$ , aside from  $Q_4$  and  $Q_9$ .

We provide the constructions in items 1 and 2 above in sections 6.1 and 6.2 respectively.

6.1. Cubes and cubes times  $K_{1,2}$ .

6.1.1. *The 8-cube.* That  $\xi_o(Q_8) = 128$  is shown by the following elegant packing of 128 tepees. In Figure 4 the vertex labels are 2-digit hexadecimal numbers. Each diagram represents four tepees, and is to be replicated eight times. The upper two by adding the span of  $\{\text{ff}, 66, 0\text{c}\}$  over  $\mathbb{F}_2$  (i.e.,  $00, \text{ff}, 66, 99, 0\text{c}, \text{f3}, 6\text{a}, 95$ ) and the lower two by adding the span of  $\{\text{ff}, 66, 03\}$  (i.e.,  $00, \text{ff}, 66, 99, 03, \text{fc}, 65, 9\text{a}$ ). The reader may verify that these exactly cover the  $8 \cdot 2^7$  edges of  $Q_8$ .

FIGURE 4.  $Q_8$  is the union of 128 tepees

6.1.2. *The 12-cube.* We exhibit an exact packing of  $Q_{12}$  establishing that  $\xi_o(Q_{12}) = 3072$ .

There is a graph homomorphism  $p$  from  $Q_{12}$  to  $Q_6$  sending the vertex  $(x, y) \in \mathbb{F}_2^6 \times \mathbb{F}_2^6$  to  $x + y \in \mathbb{F}_2^6$ . This homomorphism induces a two-to-one map from the twelve edge directions of  $Q_{12}$  to the six of  $Q_6$ , following which we grant the directions of  $Q_{12}$  the names  $0x, \dots, 5x$  and  $0y, \dots, 5y$ .

We produce our packing of  $Q_{12}$  as follows: we provide an arrangement of tepees in  $Q_6$ , and in each tepee, tag each class of edges in a single direction with the symbol  $x$  or  $y$ . Altogether each edge of  $Q_6$  will be covered by two tepee edges, one tagged  $x$  and one tagged  $y$ . This arrangement on  $Q_6$  can be pulled back across the homomorphism  $p$  to the requisite packing of  $Q_{12}$ , as suggested by the labels: a tepee with tip  $v$  and orientation  $abc$ , whose edges in directions  $a, b, c$  are tagged respectively  $\alpha, \beta, \gamma \in \{x, y\}$ , pulls back to 64 tepees with tips at each of the preimages of  $v$ , all in orientation  $\alpha\alpha\ b\beta\ c\gamma$ .

We present our arrangement on  $Q_6$  using two-digit octal numbers for the vertices. It has 48 tepees; together these have the requisite  $64 \cdot 48 = 3072$

preimages in  $Q_{12}$ . At each of the vertices 00, 11, 22, 33, 44, 55, 66, 77 are placed the tips of six tepees.

Tip location	Orientations
00	0y1x2x 1y2x3x 2y3x4x 3y4x5x 4y5x0x 5y0x1x
11	0x1x2y 1y2y3y 2x3y4x 3x4x5y 4y5y0y 5x0y1x
22	0x1y2x 1x2x3y 2y3y4y 3x4y5x 4x5x0y 5y0y1y
33	0y1y2y 1x2y3x 2x3x4y 3y4y5y 4x5y0x 5x0x1y
44	0y1y2y 1x2y3x 2x3x4y 3y4y5y 4x5y0x 5x0x1y
55	0x1y2x 1x2x3y 2y3y4y 3x4y5x 4x5x0y 5y0y1y
66	0x1x2y 1y2y3y 2x3y4x 3x4x5y 4y5y0y 5x0y1x
77	0y1x2x 1y2x3x 2y3x4x 3y4x5x 4y5x0x 5y0x1x

6.1.3. *The Cartesian product  $Q_4 \times K_{1,2}$ .* Let  $Q_4$  sit in directions 0 through 3, and label its vertices in hexadecimal; and let the directions of the two edges of  $K_{1,2}$  be 4 and 5. All of the tepees in the packing have their tips at the middle vertex of  $K_{1,2}$ , so we will only give their position in  $Q_4$ . They have

orientation 014    and    tips 0, 5, a, f;  
orientation 015    and    tips 3, 6, 9, c;  
orientation 234    and    tips 1, 4, b, e;  
orientation 235    and    tips 2, 7, 8, d.

6.1.4. *The Cartesian product  $Q_8 \times K_{1,2}$ .* We first look at a single  $Q_8$ , and label its vertices in hexadecimal. The capital letters  $ABCDEFGH$  will denote the directions or colors of edges; edges in these directions respectively have ends with difference (or, equivalently, sum) 01 02 04 08 10 20 40 80.

Consider two rings of four tepees, with trivalent vertices at 00 03 0c 0f and 30 33 3c 3f, respectively, where the third color of the tepees whose sets of trivalent vertices are  $\{00, 03\}$ ,  $\{0c, 0f\}$ ,  $\{30, 3c\}$ , and  $\{33, 3f\}$  is  $E$ , and the third color of the tepees whose sets of trivalent vertices are  $\{00, 0c\}$ ,  $\{03, 0f\}$ ,  $\{30, 33\}$ , and  $\{3c, 3f\}$  is  $F$ . The 64 edges of these tepees consist entirely of squares in the pairs of colors  $AB$ ,  $CD$ , and  $EF$ . They include the sixteen squares with pairs of opposite vertices

$$\begin{aligned} &\{00, 03\}, \{0c, 0f\}, \{10, 13\}, \{1c, 1f\}, \{30, 33\}, \{3c, 3f\}, \{00, 30\}, \{03, 33\} \\ &\{00, 0c\}, \{03, 0f\}, \{20, 2c\}, \{23, 2f\}, \{30, 3c\}, \{33, 3f\}, \{0c, 3c\}, \{0f, 3f\}. \end{aligned}$$

We place eight such groups of eight tepees in  $Q_8$ . Four of them are in the orientation displayed above, displaced by 00, 55, aa, and ff respectively. The other four are the images of the displayed tepee under the symmetries of  $Q_8$  that exchange directions  $A$  with  $G$ ,  $B$  with  $H$ ,  $C$  with  $E$ , and  $D$  with  $F$ , and map 00 to 01, 54, ab, and fe respectively.

These 64 tepees occupy half of the edges in  $Q_8$ . In any  $Q_4$  with colors  $ABCD$  or  $EF GH$ , exactly four of the squares with colors  $AB$  or  $CD$  or  $EF$  or  $GH$  have been completely filled, and the other four such squares are completely empty. Furthermore, in a  $Q_4$  of colors  $ABCD$ , each evil vertex



In our whole graph  $Q_8 \times K_{1,2}$ , we will place this configuration of 64 tepees in each of the three  $Q_8$ . This leaves us to fill the remaining half of the edges in each  $Q_8$  as well as all of the edges of the original 256 copies of  $K_{1,2}$ .

In particular, we place one of these pairs of tepees in each of the  $4 \cdot 32 = 128$  unfilled squares of colors  $AB$  or  $CD$  or  $EF$  or  $GH$  in the three subgraphs  $Q_8$ . This fills every remaining edge of these subgraphs. We place a copy of  $K_{1,2}$  at each evil vertex of a square of colors  $AB$  or  $CD$ , and at each odious vertex of a square of colors  $EF$  or  $GH$ . By our earlier observations, there's always just one square available for each vertex of the  $Q_8$ , so each  $K_{1,2}$  is filled just once. This completes the packing. In all we have used  $3 \cdot 64 + 128 \cdot 2 = 448$  tepees, as required to pack  $3 \cdot 8 \cdot 128 + 256 \cdot 2 = 3584$  edges.

**6.2. A partial  $Q_5$ .** We will pack  $Q_5$  with eight copies of the graph  $U$  shown in Figure 5 obtained from  $Q_3$  by deleting two opposite edges. The graph  $U$  accommodates a tepee in any of four ways with  $K_{1,2}$  remaining.

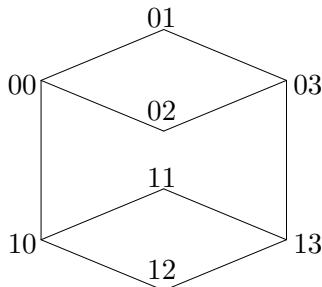


FIGURE 5. The graph  $U$

If we label the vertices of  $Q_5$  with hexadecimal labels, then the trivalent vertices of the eight copies of  $U$  can be taken to fall at

$$\begin{array}{cccc} \{00, 03\}, & \{05, 06\}, & \{0a, 09\}, & \{0f, 0c\} \\ \{11, 1d\}, & \{14, 18\}, & \{1b, 17\}, & \{1e, 12\} \end{array}$$

(these are pairs of trivalent vertices at opposite corners of a square) and the sums (over  $\mathbb{F}_2$ ) of 10 with each of these. Call these copies  $U_0, \dots, U_7$ .

It is possible to select a tepee from each  $U$  in such a way that the remaining edges form eight disconnected copies of  $K_{1,2}$ , for instance by removing from each  $U$  the two edges incident to 12,17,18,1d,09,0c,03,06 in  $U_0, \dots, U_7$  respectively.

Therefore let  $G$  be a graph such that  $G$  and  $G \times K_{1,2}$  have exact packings. In the graph  $G \times Q_5$ , each of the copies of  $Q_5$  can be packed with eight copies of  $U$ , with the unused edges positioned so as to make up eight copies each of  $G$  and  $G \times K_{1,2}$ , which have exact packings themselves. Altogether this yields an exact packing of  $G \times Q_5$ .

As we saw in item 2 of the discussion opening Section 6, this construction results in packings of all of the cube graphs  $Q_{4k+9m}$  for integers  $k, m \geq 0$  aside from  $Q_4$  and  $Q_9$ .

6.2.1. *Example: the 13-cube.* View  $Q_{13}$  as  $Q_8 \times Q_5$ . After partially filling the copies of  $Q_5$  as in Section 6.2, we are left with eight copies each of  $Q_8$  and  $Q_8 \times K_{1,2}$ , whose packings we have presented in Sections 6.1.1 and 6.1.4. Altogether this uses

$2^8 \cdot 8$  tepees in the  $2^8$  partially filled subgraphs  $Q_5$ ;  
 $2^3 \cdot 2^7$  tepees in the  $2^3$   $Q_8$  corresponding to the full vertices  
of the  $Q_5$ ;  
 $2^3 \cdot 3 \cdot 2^3 \cdot 2^3$  tepees in the  $2^3 \cdot 3$  half-filled  $Q_8$  within the  $2^3$   
 $Q_8 \times K_{1,2}$ ;  
 $2^3 \cdot 2^7 \cdot 2$  tepees in the rest of the  $Q_8 \times K_{1,2}$ ;

which totals  $13 \cdot 2^9$ .

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QUEEN MARY, UNIVERSITY OF LONDON, LONDON, ENGLAND  
*E-mail address*, Alex Fink: `a.fink@qmul.ac.uk`

THE UNIVERSITY OF CALGARY, CALGARY ALBERTA, CANADA  
*E-mail address*, Richard K. Guy: `rk@cpsc.ucalgary.ca`